The Synthesis of *N*-Port Circulators*

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Summary—It is shown that there is a certain wide class of three-port network that can be transformed into a circulator by the addition of reactive elements. For such synthesis from a symmetrical three-port junction it is necessary and sufficient for it to be loss-free and for the moduli of the two transmission coefficients to be different. This approach may be used in the design of broad-band circulators.

For junctions approximating to an n-port circulator with $n>3$, it is shown that when the n reflection coefficients are matched by reactive networks in each arm, n particular transmission coefficients also vanish.

I. Introduction

The problems of circulator design have been considered both experimentally and theoretically for several years.[1,2] Of particular current interest are the symmetrical 3- and 4-port circulators employing either waveguide or strip-line.[3,4] This present contribution is, however, not restricted to any physical form of junction, but applies to general loss-free n-port junctions. In the analysis we study the effect of placing a reactive 2-port network in series with each arm of an n-port nonreciprocal loss-free junction.

II. Analysis

If S is the scattering matrix of a loss-free n-port junction then S is unitary, *viz.*,

$$SS^* = 1 = S^*S \qquad (1)$$

where S^* is the conjugate transpose of S.

The scattering matrix of a symmetrical reactive two-port network is of the form

$$T = \begin{pmatrix} \rho & t \\ t & \rho \end{pmatrix}.$$

This matrix also is unitary, giving the relations

$$\begin{aligned} \rho\rho^* + tt^* &= 1 \\ \rho t^* + \rho^* t &= 0. \end{aligned} \qquad (2)$$

If a_i and b_i are the incident and reflected wave amplitudes at the ith port of the original n-port junction and if a_i' and b_i' are the corresponding amplitudes for the junction with reactive networks in series with each port (see, for instance, Fig. 1 where $n=3$), then for each i we have

$$\begin{pmatrix} b_i' \\ a_i \end{pmatrix} = \begin{pmatrix} \rho_i & t_i \\ t_i & \rho_i \end{pmatrix} \begin{pmatrix} a_i' \\ b_i \end{pmatrix}.$$

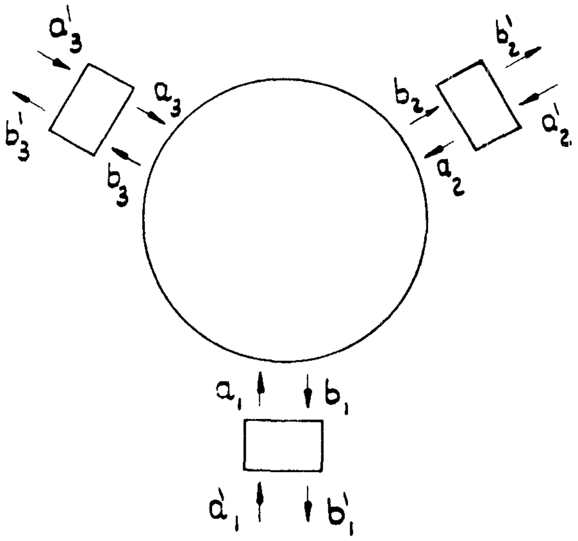

Fig. 1.

Therefore, omitting the suffix

$$a = (t^2 - \rho^2)a'/t + \rho b/t$$

$$b = -\rho a'/t + b/t$$

and by (2)

$$\left.\begin{aligned} a &= \frac{a'}{t^*} + \frac{\rho}{t}b' \\ b &= \frac{\rho^*}{t^*}a' + \frac{b'}{t} \end{aligned}\right\}. \qquad (3)$$

Now

$$\mathbf{b} = S\mathbf{a}$$

where $\mathbf{a}$ and $\mathbf{b}$ denote the column vectors with elements a_i and b_i. Substituting from (3) gives

$$(Q^*\mathbf{a}' + P\mathbf{b}') = S(P^*\mathbf{a}' + Q\mathbf{b}')$$

where P and Q are the diagonal matrices

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[2] B. A. Auld, "The synthesis of symmetrical waveguide circulators," IRE Trans. on Microwave Theory and Techniques, vol. MTT-7, pp. 238–246; April, 1959.

[3] H. Bosma, "On the principle of stripline circulation," *Proc. IEE*, B, vol. 109, Supplement No. 21, p. 137–146; January, 1962.

[4] J. B. Davies, "An analysis of the m-port symmetrical H-plane waveguide junction with central ferrite post," this issue, pp. 596–604.

$$P = \begin{bmatrix} 1/t_1 & 0 & \cdots & 0 \\ 0 & 1/t_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1/t_n \end{bmatrix} \tag{4}$$

$$Q = \begin{bmatrix} \rho_1/t_1 & 0 & \cdots & 0 \\ 0 & \rho_2/t_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \rho_n/t_n \end{bmatrix}. \tag{5}$$

Therefore

$$(P - SQ)\boldsymbol{b}' = (SP^* - Q^*)\boldsymbol{a}'.$$

If R is the resultant scattering matrix

$$\boldsymbol{b}' = R\boldsymbol{a}',$$

and therefore

$$(P - SQ)R = (SP^* - Q^*). \tag{6}$$

We wish now to consider conditions under which the matrices P and Q may exist for any S, such that

$$R = \begin{bmatrix} 0 & e^{i\theta_1} & 0 & \cdots \\ 0 & 0 & e^{i\theta_2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ e^{i\theta_n} & 0 & 0 & \cdots 0 \end{bmatrix} \tag{7}$$

corresponding to an n-port circulator.

The general n-port lossless junction is specified by n^2 real numbers. By inserting symmetrical reciprocal two-port networks in series with each of the n ports, $2n$ real numbers can be chosen, in addition to the n phase angles of R. Clearly a necessary condition for the synthesis of a circulator by this approach is that $n \leq 3$.

A. Synthesis from the General Loss-Free 3-Port Junction

The scattering matrix of a three-port junction may be written

$$S = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \gamma_2 & \alpha_2 & \beta_2 \\ \beta_3 & \gamma_3 & \alpha_3 \end{bmatrix} \tag{8}$$

and from (1) the following relationships may be obtained:

$$\left.\begin{aligned} \alpha_i\alpha_i^* + \beta_i\beta_i^* + \gamma_i\gamma_i^* &= 1 \\ \alpha_i\alpha_i^* + \gamma_j\gamma_j^* + \beta_k\beta_k^* &= 1 \\ \alpha_i\gamma_j^* + \beta_i\alpha_j^* + \gamma_i\beta_j^* &= 0 \\ \alpha_i\beta_i^* + \gamma_j\alpha_j^* + \beta_k\gamma_k^* &= 0 \end{aligned}\right\} \tag{9}$$

The subscripts i, j and k are a cyclic permutation of 1, 2 and 3. Clearly the conjugate form of the last two equations can also be taken.

Substituting (4), (5), (7) and (8) into (6) and equating the corresponding elements gives

$$-\rho_1 \exp(i\theta_1)\frac{t_2^*}{t_1} = \frac{\beta_1\rho_1}{(\rho_1\alpha_1 - 1)} = \frac{(\alpha_2 - \rho_2^*)}{\gamma_2} = \frac{\gamma_3}{\beta_3} \tag{10}$$

$$-\rho_2 \exp(i\theta_2)\frac{t_3^*}{t_2} = \frac{\beta_2\rho_2}{(\rho_2\alpha_2 - 1)} = \frac{(\alpha_3 - \rho_3^*)}{\gamma_3} = \frac{\gamma_1}{\beta_1} \tag{11}$$

$$-\rho_3 \exp(i\theta_3)\frac{t_1^*}{t_3} = \frac{\beta_3\rho_3}{(\rho_3\alpha_3 - 1)} = \frac{(\alpha_1 - \rho_1^*)}{\gamma_1} = \frac{\gamma_2}{\beta_2}. \tag{12}$$

ρ_1 can be chosen to satisfy

$$\rho_1 = \frac{\gamma_3}{(\gamma_3\alpha_1 - \beta_1\beta_3)} \tag{13}$$

from (10), as can ρ_2 and ρ_3 similarly. These values will now be shown to satisfy all of (10), (11) and (12).

Substituting for ρ_1 in (12) gives

$$(\gamma_3\alpha_1 - \beta_1\beta_3)(\alpha_1^*\beta_2^* - \gamma_1^*\gamma_2^*) - \gamma_3\beta_2^* = 0$$

and using the unitary condition, (9)

$$\gamma_3\beta_2^*(\beta_1\beta_1^* + \gamma_1\gamma_1^*) + \alpha_1\gamma_1^*\gamma_2^*\gamma_3 + \alpha_1^*\beta_2^*\beta_1\beta_3 - \beta_1\beta_3\gamma_1^*\gamma_2^* = 0$$

$$\therefore\ \gamma_3\gamma_1^*(\gamma_1\beta_2^* + \alpha_1\gamma_2^*) + \gamma_3\beta_2^*\beta_1\beta_1^* + \alpha_1^*\beta_1\beta_2^*\beta_3 - \beta_1\beta_3\gamma_1^*\gamma_2^* = 0$$

$$\therefore\ -\gamma_3\gamma_1^*\alpha_2^*\beta_1 - \beta_1\beta_2^*\gamma_1^*\alpha_3 - \beta_1\beta_3\gamma_1^*\gamma_2^* = 0$$

$$\therefore\ -\gamma_1^*\beta_1(\gamma_3\alpha_2^* + \alpha_3\beta_2^* + \beta_3\gamma_2^*) = 0.$$

But from (9)

$$\gamma_3\alpha_2^* + \alpha_3\beta_2^* + \beta_3\gamma_2^* = 0.$$

Hence ρ_1, as given by (13), satisfies (12). Similarly (10) and (11) are satisfied by the corresponding values of ρ_2 and ρ_3.

It will now be sufficient to prove that ρ_1 given by (13) satisfies

$$\left|\rho_1\frac{t_2^*}{t_1}\right| = \left|\frac{\gamma_3}{\beta_3}\right|$$

and similarly for ρ_2 and ρ_3. Now $\rho\rho^* + tt^* = 1$ and so $\rho_1\rho_1^*\, t_2t_2^*/t_1t_1 = \gamma_3\gamma_3^*/\beta_3\beta_3^*$ is equivalent to

$$\beta_3\beta_3^*\rho_1\rho_1^* - \beta_3\beta_3^*\rho_1\rho_1^*\rho_2\rho_2^* = \gamma_3\gamma_3^* - \rho_1\rho_1^*\gamma_3\gamma_3^*. \tag{14}$$

But from (13)

$$\rho_1\rho_2^* = \frac{\gamma_3}{(\gamma_3\alpha_1 - \beta_1\beta_3)} \times \frac{\gamma_1^*}{(\gamma_1^*\alpha_2^* - \beta_2^*\beta_1^*)}.$$

Therefore,

$$\frac{\gamma_3\gamma_1^*}{\rho_1\rho_2^*} = \gamma_3\alpha_1\gamma_1^*\alpha_2^* - \beta_1\beta_3\gamma_1^*\alpha_2^* - \beta_2^*\beta_1^*\gamma_3\alpha_1 + \beta_1\beta_1^*\beta_2^*\beta_3$$
$$= \gamma_1^*\gamma_3\alpha_1\alpha_2^* + \beta_2^*\gamma_3(\gamma_2\alpha_2^* + \beta_3\gamma_3^*) + \beta_3\alpha_2^*(\alpha_2\beta_2^* + \gamma_3\alpha_3^*) + \beta_2^*\beta_3(1 - \alpha_2\alpha_2^* - \gamma_3\gamma_3^*)$$
$$= \gamma_1^*\gamma_3\alpha_1\alpha_2^* + \beta_2^*\gamma_3\gamma_2\alpha_2^* + \beta_3\alpha_2^*\gamma_3\alpha_3^* + \beta_2^*\beta_3$$
$$= \gamma_3\alpha_2^*(\alpha_1\gamma_1^* + \gamma_2\beta_2^* + \beta_3\alpha_3^*) + \beta_2^*\beta_3$$
$$= \beta_2^*\beta_3.$$

Eq. (14) becomes

$$\beta_3\beta_3^*\rho_1\rho_1^* - \beta_3\beta_3^* \frac{\gamma_3\gamma_1^*}{\beta_2^*\beta_3}\cdot\frac{\gamma_3^*\gamma_1}{\beta_2\beta_3^*} = \gamma_3\gamma_3^* - \rho_1\rho_1^*\gamma_3\gamma_3^*$$

or

$$\rho_1\rho_1^*(\beta_3\beta_3^* + \gamma_3\gamma_3^*) = \frac{\gamma_3\gamma_3^*}{\beta_2\beta_2^*}(\gamma_1\gamma_1^* + \beta_2\beta_2^*)$$

or

$$\rho_1\rho_1^* = \frac{\gamma_3}{\beta_2}\cdot\frac{\gamma_3^*}{\beta_2^*}. \tag{15}$$

Again substituting from (13)

$$\beta_2\beta_2^* = (\gamma_3\alpha_1 - \beta_1\beta_3)(\gamma_3^*\alpha_1^* - \beta_1^*\beta_3^*)$$

or

$$\beta_2\beta_2^* - \gamma_3\gamma_3^*\alpha_1\alpha_1^* + \gamma_3^*\alpha_1^*\beta_1\beta_3 + \gamma_3\alpha_1\beta_1^*\beta_3^* - \beta_1\beta_1^*\beta_3\beta_3^* = 0.$$

Using the unitary condition of (9), we have

$$\beta_2\beta_2^* - \gamma_3\gamma_3^*\alpha_1\alpha_1^* - \gamma_3^*\beta_1(\beta_1^*\gamma_3 + \gamma_1^*\alpha_3) - \alpha_1\beta_3^*(\beta_3\alpha_1^* + \alpha_3\gamma_1^*) - \beta_1\beta_1^*\beta_3\beta_3^*$$
$$= \beta_2\beta_2^* - \gamma_3\gamma_3^*(\alpha_1\alpha_1^* + \beta_1\beta_1^*) - \alpha_3\beta_1\gamma_1^*\gamma_3^* - \alpha_1\alpha_3\beta_3^*\gamma_1^* - \beta_3\beta_3^*(\alpha_1\alpha_1^* + \beta_1\beta_1^*)$$
$$= \beta_2\beta_2^* - (\alpha_1\alpha_1^* + \beta_1\beta_1^*)(\gamma_3\gamma_3^* + \beta_3\beta_3^*) + \alpha_3\gamma_1^*\gamma_1\alpha_3^*$$
$$= \beta_2\beta_2^* - (1 - \gamma_1\gamma_1^*)(1 - \alpha_3\alpha_3^*) + \alpha_3\alpha_3^*\gamma_1\gamma_1^*$$
$$= \beta_2\beta_2^* + \gamma_1\gamma_1^* + \alpha_3\alpha_3^* - 1 = 0.$$

Therefore (15) and (14) are satisfied, and we have proved (13) to give a consistent solution to (6), the circulation condition.

We note from (15) that because $|\rho| < 1$ for passive networks, we must have

$$\left.\begin{aligned} |\gamma_3| &< |\beta_2| \\ |\gamma_1| &< |\beta_3| \\ |\gamma_2| &< |\beta_1| \end{aligned}\right\}. \tag{16}$$

Clearly if the operator R had been taken in the opposite sense, the restriction would be

$$\left.\begin{aligned} |\gamma_3| &> |\beta_2| \\ |\gamma_1| &> |\beta_3| \\ |\gamma_2| &> |\beta_1| \end{aligned}\right\}. \tag{17}$$

When the original three-port junction is symmetrical, our restriction reduces to the coefficients satisfying $|\gamma| \neq |\beta|$.

Although the reflection coefficients of the required discontinuities have been derived exactly, it will be noted that a convenient approximation can be obtained from (10)–(12) when the three-port network already has characteristics which are approximately those of a circulator. The reflection coefficient of each two-port network is then the complex conjugate (α_1^*, α_2^* or α_3^*) of the respective reflection coefficient of the original three-port network.

B. Synthesis from the General N-Port Junction

We have already demonstrated that the n-port circulator cannot be synthesized from the general n-port junction if $n \geq 4$. However, of practical interest is the case in which the n-port junction already approximates to a circulator. Suppose the two-port networks have reflection coefficients equal to the complex conjugates of the corresponding reflection coefficients of the original n-port junction. Then (6), which can be written

$$R = \{(P - SQ)^{-1} - P^*\}Q^{-1},$$

leads to the expansion

$$R = P^{-1}\{S - T + STS + O(T^2)\}P^{-1} = P^{-1}XP^{-1} \tag{18}$$

where $T = QP^{-1} = \text{Diagonal } (S^*)$.

The multiplications by P^{-1} of (18) leave substantially unaltered the magnitude of each element of the matrix X.

If now at each port the two-port network matches the reflection coefficient, it will be shown that one transmission coefficient corresponding to each port also vanishes.

If $(S)_{ij} = s_{ij}$ and $(X)_{ij} = x_{ij}$, (18) gives the following first order relations:

$$x_{ii} = \sum_{k=1}^{n} s_{kk}^* s_{ki} s_{ik} \tag{19}$$

$$x_{ij} = s_{ij} + \sum_{k=1}^{n} s_{kk}^* s_{ik} s_{kj}. \tag{20}$$

Since the original n-port junction approximates to a circulator, we can take the ports to be numbered such that $|s_{i,i+1}| \approx 1$. Eqs. (19) and (20) become

$$x_{ii} = 0$$
$$x_{ij} = s_{ij} \text{ for } j \neq i \text{ or } i+2$$
$$x_{i-1,i+1} = s_{i-1,i+1} + s_{ii}^* s_{i-1,i} s_{i,i+1}. \tag{21}$$

From the unitary condition on S, we have

$$(S^*S)_{i,i+1} = \sum_{k=1}^{n} s_{k,i}s_{k,i+1} = 0$$

and therefore $s_{i-1,i}^* \, s_{i-1,i+1} + s_{i,i}^* \, s_{i,i+1} = 0$

Substituting in (21) gives

$$x_{i-1,i+1} = 0 \quad \text{for all } i.$$

Hence the matching at port i isolates port $i+1$ from port $i-1$.

III. Conclusions

It has been shown that a certain class of three-port network can be transformed into a circulator by the addition of an appropriate reactive discontinuity at each port. This transformation is possible if the three-port junction is loss-free and such that (for suitable port numbering) the transmission from port 1 to 2 is greater than from 2 to 1, from port 2 to 3 is greater than 3 to 2, and from 3 to 1 is greater than from 1 to 3. When the three-port network is symmetrical, the restriction reduces to the network, being loss-free and the moduli of the two transmission coefficients being different.

The important consequence of the proposed synthesis is that it is not necessary to obtain complete matching by means of the ferrite junction configuration and applied magnetic field, but that external reciprocal elements may be used, which have, of course, predictable characteristics. Hence, an approach to the design of broad-band circulators is proffered.

It has also been shown that by the use of small reactive discontinuities, an imperfect (but loss-free) n-port circulator can be matched to make vanish, simultaneously, the reflection and one transmission coefficient at each port. The effect of these small discontinuities on the other transmission coefficients is of second order of smallness, so that they cannot generally be made zero. It follows that the isolation of a practical four-port circulator must be optimized within the junction, but any small remaining reflection and cross-coupling can be matched externally by reactive discontinuities.

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Propagation of Surface Waves on an Inhomogeneous Plane Layer*

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Summary—**The permittivity of a plane layer is assumed to vary continuously as a function of distance measured from the surface. Solutions for the field distributions of surface waves on the inhomogeneous layer are developed with the WKB technique. Transcendental equations for the phase velocity are derived for TE and TM modes. These equations are solved most conveniently with the aid of phase-velocity graphs which are included. The accuracy of the solution is verified by comparison with the rigorous solution for an exponential inhomogeneity.**

Introduction

The PERMITTIVITY of most radome materials changes by a significant amount when the temperature is increased by hypersonic flight through the atmosphere. The outer surface of the radome becomes hotter than the inner, resulting in a continuous variation in permittivity even if the radome was designed as a homogeneous structure.

Moreover, new techniques of radome fabrication may make feasible the construction of continuously inhomogeneous radomes. This can be accomplished with variable loading or with variable density foams. Alternatively, a multilayer sandwich having many thin laminations can form an adequate approximation. These structures may have a greater bandwidth or may allow a greater range of incidence angles than conventional radomes.

The characteristics of surface waves on inhomogeneous layers are of interest to the radome designer because he must minimize the excitation of these waves and their deletereous effects on the radar system performance. The antenna designer is interested in the effects of unintentional inhomogeneities, arising from thermal gradients, on the performance of surface-wave antennas, and the advantages that may accrue from the use of intentional inhomogeneities in such antennas.

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